MTH 304: Metric Spaces and Topology

Practice Assignment II: Connectedness

1. Establish the following assertions in 1.11 of the lesson plan: (iv), (viii) (b) & (c), (xi) (b), (xv), (xxi), and (xxiv).

2. Is the topologist's sine curve locally path connected? Why or why not?

3. Using the properties of connected or path connected spaces, establish the following.

   (a) The interval (a, b), $(a, b]$, and $[a, b]$ are not homeomorphic to each other?

   (b) $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$, for any $n > 1$.

   (c) For a continuous map $f : S^1 \to \mathbb{R}$, there exists a point $x \in S^1$ such that $f(x) = f(-x)$.

4. Prove or disprove (by giving an appropriate counterexample), the following statements.

   (a) The product of path-connected spaces is path-connected.

   (b) A locally connected space is locally path connected.

   (c) If $A \subset X$ is path connected, then $\bar{A}$ is path connected.

   (d) An open connected subspace of $\mathbb{R}^n$ is path connected.

   (e) If $A \subset X$ is connected, then A° and ∂A are connected.

   (f) Every open connected subset of a locally path connected space is path connected.

   (g) Image of a locally connected space under a quotient map is locally connected.

5. A topological space X is said to be *totally disconnected* if the only connected subspaces of X are the singletons and $\emptyset$. Establish the following.

   (a) Subspaces and products of totally disconnected spaces are totally disconnected.

   (b) Every totally disconnected space satisfies the T_1 axiom.

   (c) Continuous images of totally disconnected spaces are not necessarily totally disconnected.

   (d) A discrete topological space is totally disconnected.

   (e) The Cantor set ($\subset [0, 1]$), $\mathbb{Q}$, and $\mathbb{R} \setminus \mathbb{Q}$ are totally disconnected spaces.

6. Describe the path components of $\mathbb{R}^\infty$ in product topology. Also, describe its components in the box and uniform topologies.